Analysis of the Nonlinear Vibrations of Unsymmetrically Laminated Composite Beams

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Large-amplitude free vibrations of unsymmetrically laminated beams using von Karman large deflection theory are investigated herein. One-dimensional finite elements based on classical lamination theory, first-order shear-deformation theory, and higher-order shear-deformation theory having 8, 10, and 12 degrees of freedom per node, respectively, are developed to bring out the effects of transverse shear on the large-amplitude vibrations. Because of the presence of bending-extension coupling, the bending stiffness of an unsymmetric laminate is direction dependent yielding different amplitudes and spatial deformations for the positive and negative deflection half-cycles. The problem is studied by reducing the dynamic nonlinear finite element equations to two second-order ordinary nonlinear differential equations using converged normalized spatial deformations in the positive and negative deflection half-cycles. These modal equations of motion are solved using the direct numerical integration method and results are presented for various boundary conditions, lay-up sequences, and slenderness ratios. Inadequacies in the results of approximate methods are highlighted.

Nomenclature

A,B,D,E,F,H	= extensional, bending-extension coupling, bending, and high-order stiffness coefficients
A_{max}, B_{max}	= amplitude of positive and negative deflection half-cycles, respectively
B_o, B_L	= matrices relating strains to nodal degrees of freedom
E_L, E_T, G_{LT}	= in-plane Young's moduli in fiber and transverse direction and shear modulus
G_{LZ}	= transverse shear modulus
h	= element length
N, M, P	= stress, moment, and higher-order stress resultants
N_i	= shape functions
q_i	= eigenvector
T	= kinetic energy
U	= strain energy
u, v, w	= displacements in the x, y, and z directions, respectively
u_o, v_o, w_o	= midplane displacements in the x, y, and z directions, respectively
$\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{xyo}$	= midplane strains
$\varkappa_{xo}, \varkappa_{yo}, \varkappa_{xyo}$	= midplane curvatures
$\varkappa_{x1}, \varkappa_{y1}, \varkappa_{xy1}$	= curvatures corresponding to shear deformations
τ	= $w_{,y}$ (twist)
α_1, β_1, γ_1, α_2, β_2, γ_2	= stiffness coefficients of linear, quadratic, and cubic terms of the modal equation of equilibrium for positive and negative deflection half-cycles
ν_{LT}	= Poisson's ratio

Introduction

COMPOSITES are a class of materials that have revolutionized practically every sphere of technology during the last three decades, because of their prime importance to the technological progress in the area of aerospace and aircraft industry. However, the analysis of composite structures is a complex task due to the bending-extension coupling. These structures (beams, plates, and shells) are very often subjected to a severe dynamic environment necessitating the need for their large-amplitude vibration behavior. Furthermore, it has been pointed out by numerous researchers that since the composites have very low transverse shear modulus compared to their in-plane moduli, the classical lamination theory (CLT) will lead to the upper bound of stiffness. As a consequence, CLT will underpredict the deflections and overpredict the frequencies and buckling loads.

Sathyamoorthy[1,2] presented a comprehensive survey of large-amplitude free vibrations of beams using approximate analytical methods and numerical methods (finite elements). The text book by Chia[3] provides a lot of information on nonlinear response of composite plates. Miller and Adam[4] and Teoh and Huang[5] have studied the vibrations of composite beams. Wang[6] investigated the vibrations of channel-sectioned laminated composite beams using an analytical approach. Besides Refs. 4–6 the vibrations of composite beams using first-order shear deformation theory are investigated by Abarcar and Cuniff,[7] Murty and Shimpi,[8] Chen and Yang,[9] Teh and Huang,[10] Whitney et al.,[11] and Kapania and Raciti.[12] Higher-order shear deformation theories have been developed by Reddy,[13–15] Kromm,[16] and Lo et al.[17] to study the bending behavior of composite structures. Singh et al.[18,19] investigated the nonlinear vibrations of beams and antisymmetric cross-ply plates using the direct numerical integration method and found that the perturbation method used by Chandra and Raju[20,21] fails for laminates having predominant bending-extension coupling.

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Kapania and Raciti[12] have studied the nonlinear vibrations of composite beams. They developed a composite beam element based on first-order shear-deformation theory to investigate the large-amplitude vibrations. The dynamic finite element equations were reduced to a single second-order ordinary nonlinear differential equation using the linear mode shape of vibration, which is solved using the perturbation method. Furthermore, they have included a term $(hb^3/12)\dot{\tau}^2$ in the kinetic energy which is not consistent with the displacement field assumed. Atluri[22] has shown that, for hinged beams with movable ends, the nonlinearity is mainly due to curvatures $[\theta_{,x} = w_{,xx}/(1 - w_{,x})^{0.5}]$ leading to a soft-spring type of behavior rather than the axial stretching as is considered in Ref. 12.

In the present paper, three composite beam elements based on the classical lamination theory (CLT), first-order shear-deformation theory (FSDT), and higher-order shear-deformation theory (HSDT) are developed to investigate the large-amplitude vibrations. In the HSDT, the displacement field assumed is such that if the shear terms corresponding to the higher power of thickness are dropped, the CLT is obtained. The dynamic finite element equations are solved iteratively with harmonic oscillations assumption for both the positive deflection half-cycle and negative deflection half-cycle. The converged normalized spatial deformations for the positive and negative deflection half-cycles are used to obtain two ordinary second-order differential equations governing the motion in the positive and negative half-cycles, respectively. These equations are solved using the direct numerical integration method[18,19] to obtain the nonlinear periods/frequencies of the unsymmetric composite laminates. The method yields very accurate results as shown in Refs. 18 and 19. The effects of slenderness ratio, boundary conditions, and lay-up are discussed in the present paper.

Formulation

Consider a beam length l, width b, and total thickness t, made up of a number of perfectly bonded layers as shown in Fig. 1. Each lamina made of unidirectional fiber-reinforced material is considered as homogeneous orthotropic. The orthotropic axes of symmetry in each lamina of arbitrary thickness and elastic properties are oriented at an arbitrary angle θ to the beam axis.

Beam elements based on the classical lamination theory, first-order shear-deformation theory and higher-order shear-deformation theory having 8, 10, 12 degrees of freedom per node, respectively, are developed as follows.

Classical Lamination Theory

The classical lamination theory is a direct extension of classical plate theory based on Kirchoff's hypothesis for homogeneous plates. This theory is adequate when the slenderness ratio of the beam is large. The following displacement field can be assumed:

$$u(x,y,z) = u_o(x,y) - zw_{,x} \tag{1a}$$

$$v(x,y,z) = v_o(x,y) - zw_{,y} \tag{1b}$$

$$w(x,y,Z) = w_o(x,y) \tag{1c}$$

The nonlinear strain-displacement relations of the von Karman type for laminate can be written as

$$\varepsilon_x = u_{o,x} + z\varkappa_x + \tfrac{1}{2}w_{,x}^2 \tag{2a}$$

$$\varepsilon_y = v_{o,y} + z\varkappa_y + \tfrac{1}{2}w_{,y}^2 \tag{2b}$$

$$\begin{aligned}\gamma_{xy} &= u_{o,y} + v_{o,x} + 2z\varkappa_{xy} + w_{,x}w_{,y}\\ &= \beta - 2z\tau_{,x} + \tau w_{,x}\end{aligned} \tag{2c}$$

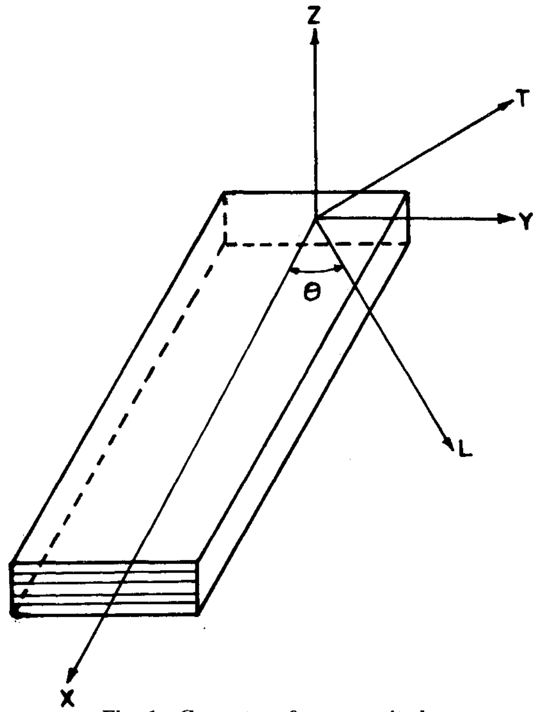

Fig. 1 Geometry of a composite beam.

The constitutive relation for a composite laminate can be written in the usual way as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \varkappa \end{Bmatrix} \tag{3}$$

In general, in the case of beam Ny and My can be equated to zero to get expressions for ε_{yo} and $\varkappa_y$ (assumed nonzero) in terms of ε_{xo}, γ_{xyo}, $\varkappa_x$, and $\varkappa_{xy}$ as follows:

$$\begin{aligned}\varepsilon_y = \Bigg\{&\left(A_{21} - \frac{B_{22}B_{21}}{D_{22}}\right)\varepsilon_{xo} + \left(A_{26} - \frac{B_{22}B_{26}}{D_{22}}\right)\gamma_{xyo}\\ &+ \left(B_{21} - \frac{B_{22}D_{21}}{D_{22}}\right)\varkappa_x + \left(B_{26} - \frac{B_{22}D_{26}}{D_{22}}\right)\varkappa_{xy}\Bigg\}\\ &\Bigg/\left(\frac{B_{22}B_{22}}{D_{22}} - A_{22}\right)\end{aligned} \tag{4a}$$

$$\begin{aligned}\varkappa_y = \Bigg\{&\left(B_{21} - \frac{B_{22}A_{21}}{A_{22}}\right)\varepsilon_{xo} + \left(B_{26} - \frac{B_{22}A_{26}}{A_{22}}\right)\gamma_{xyo}\\ &+ \left(D_{21} - \frac{B_{22}B_{21}}{A_{22}}\right)\varkappa_x + \left(D_{26} - \frac{B_{22}B_{26}}{A_{22}}\right)\varkappa_{xy}\Bigg\}\\ &\Bigg/\left(\frac{B_{22}B_{22}}{A_{22}} - D_{22}\right)\end{aligned} \tag{4b}$$

Using Eqs. (3) and (4), the constitutive relations are modified as

$$\begin{aligned}\begin{Bmatrix} Nx \\ Nxy \\ Mx \\ Mxy \end{Bmatrix} &= \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} \\ A_{16} & A_{66} & B_{16} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{16} \\ B_{16} & B_{66} & D_{16} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_x \\ \varkappa_{xy} \end{Bmatrix}\\ &+ \begin{bmatrix} A_{12} & B_{12} \\ A_{26} & B_{26} \\ B_{12} & D_{12} \\ B_{26} & D_{26} \end{bmatrix} \begin{Bmatrix} \varepsilon_y \\ \varkappa_y \end{Bmatrix}\end{aligned} \tag{5}$$

or

$$\begin{Bmatrix} Nx \\ Nxy \\ Mx \\ Mxy \end{Bmatrix} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{13} & \overline{D}_{14} \\ \overline{D}_{21} & \overline{D}_{22} & \overline{D}_{23} & \overline{D}_{24} \\ \overline{D}_{31} & \overline{D}_{32} & \overline{D}_{33} & \overline{D}_{34} \\ \overline{D}_{41} & \overline{D}_{42} & \overline{D}_{43} & \overline{D}_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_x \\ \varkappa_{xy} \end{Bmatrix} \quad (6)$$

The field variables u_o, β, w_o, and τ for finite element formulation are assumed as

$$u_o = \sum_{i=1}^{4} N_i u_i, \qquad \beta = \sum_{i=1}^{4} N_i \beta_i$$

$$w_o = \sum_{i=1}^{4} N_i w_i, \qquad \tau = \sum_{i=1}^{4} N_i \tau_i \quad (7)$$

Substituting Eq. (7) into Eq. (2), the strain-displacement relations become

$$\begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_x \\ \varkappa_{xy} \end{Bmatrix} = \left[B_o + \tfrac{1}{2} B_L(q_i) \right] \{q_i\} \quad (8)$$

where

$$\{q_i\} = [u_i u,_{xi} \beta_i \beta,_{xi} w_i w,_{xi} \tau_i \tau,_{xi}]$$

$$[B_o] = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 \\ 0 & 0 & -N_{i,xx} & 0 \\ 0 & 0 & 0 & -2N_{i,x} \end{bmatrix}$$

Similarly using the standard procedure,[24] $[B_L]$ becomes

$$[B_L] = \begin{bmatrix} 0 & 0 & w,_x N_{i,x} & 0 \\ 0 & 0 & \tau N_{i,x} & w,_x N_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

First-Order Shear-Deformation Theory

In the first-order shear-deformation theory (FSDT), the assumption that midplane normals remain normal after deformation is relaxed to midplane normals remaining straight after deformation and need not be normal. The following displacement field is assumed in this case as

$$u(x,y,z) = u_o(x,y) - z w_{b,x} \quad (9a)$$

$$v(x,y,z) = v_o(x,y) - z w_{b,y} \quad (9b)$$

$$w(x,y,Z) = w_b(x,y) + w_s(x,y) \quad (9c)$$

The nonlinear strain-displacement relations of the von Karman type can be written as

$$\varepsilon_x = \varepsilon_{xo} + z\varkappa_x = u_{o,x} + \tfrac{1}{2} w_{b,x}^2 - z w_{b,xx} \quad (10a)$$

$$\varepsilon_y = \varepsilon_{yo} + z\varkappa_y = v_{o,y} + \tfrac{1}{2} w_{b,y}^2 - z w_{b,yy} \quad (10b)$$

$$\gamma_{xy} = \gamma_{xyo} + 2z\varkappa_{xy} = u_{o,y} + v_{o,x} + w_{b,x} w_{b,y}$$
$$- 2z w_{b,xy} = \beta + \tau w_{b,x} - 2z\tau,_x \quad (10c)$$

$$\gamma_{xz} = u,_z + w,_x = w_{s,x} \quad (10d)$$

The constitutive relations are modified as in the case of the CLT, by adding an additional stiffness coefficient corresponding to the transverse shear modulus as

$$\begin{Bmatrix} Nx \\ Nxy \\ Mx \\ Mxy \\ Q_x \end{Bmatrix} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{19} & \overline{D}_{14} & 0 \\ \overline{D}_{21} & \overline{D}_{22} & \overline{D}_{29} & \overline{D}_{24} & 0 \\ \overline{D}_{31} & \overline{D}_{32} & \overline{D}_{33} & \overline{D}_{34} & 0 \\ \overline{D}_{41} & \overline{D}_{42} & \overline{D}_{43} & \overline{D}_{44} & 0 \\ 0 & 0 & 0 & 0 & \overline{D}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_x \\ \varkappa_{xy} \\ \gamma_{xz} \end{Bmatrix} \quad (11)$$

where

$$\overline{D}_{55} = k_4 \sum_{i=1}^{N} Q_{44}^i (h_{i+1} - h_i)$$

with k_4 being the shear correction factor (assumed to be 5/6).

The field variables u_o, β, w_b, w_s, and τ for the finite element formulation are assumed as in the previous case; that is,

$$u_o = \sum_{i=1}^{4} N_i u_i, \qquad \beta = \sum_{i=1}^{4} N_i \beta_i \quad (12a)$$

$$w_b = \sum_{i=1}^{4} N_i w_{bi}, \qquad \tau = \sum_{i=1}^{4} N_i \tau_i \quad (12b)$$

$$w_s = \sum_{i=1}^{4} N_i w_{si} \quad (12c)$$

Substituting Eq. (12) into Eq. (10), the strain-displacement relations in terms of nodal degrees of freedom can be written as

$$\begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_x \\ \varkappa_{xy} \\ \gamma_{xz} \end{Bmatrix} = \left[B_o + \tfrac{1}{2} B_L(q_i) \right] \{q_i\} \quad (13)$$

where

$$\{q_i\} = [u_i\ u,_{xi}\ \beta_i\ \beta,_{xi}\ w_{bi}\ w_{b,xi}\ w_{si}\ w_{s,xi}\ \tau_i\ \tau,_{xi}]$$

$$[B_o] = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & -N_{i,xx} & 0 & 0 \\ 0 & 0 & 0 & -2N_{i,x} & 0 \\ 0 & 0 & 0 & 0 & N_{i,x} \end{bmatrix}$$

$$[B_L] = \begin{bmatrix} 0 & 0 & w,_x N_{i,x} & 0 & 0 \\ 0 & 0 & \tau N_{i,x} & 0 & w,_x N_i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Higher-Order Shear-Deformation Theory

The higher-order shear-deformation theory (HSDT) not only includes transverse shear as in the case of the FSDT, but also accounts for a parabolic variation of transverse shear through the laminate thickness, and hence there is no need to use the shear correction factor as in the FSDT. The following displacement field is chosen:

$$u(x,y,z) = u_o(x,y) - z w_{b,x} - \frac{4z^3}{3h^2} w_{s,x} \quad (14a)$$

$$v(x,y,z) = v_o(x,y) - z w_{b,y} - \frac{4z^3}{3h^2} w_{s,y} \quad (14b)$$

$$w(x,y,z) = w_b(x,y) + w_s(x,y) \tag{14c}$$

The nonlinear strain-displacement relations of von Karman type can written as

$$\varepsilon_x = \varepsilon_{xo} + z\varkappa_{xo} + z^3\varkappa_{x1} = u_{o,x} + \tfrac{1}{2}w_{b,x}^2 - zw_{b,xx} - \frac{4z^3}{3h^2}w_{s,xx} \tag{15a}$$

$$\varepsilon_y = \varepsilon_{yo} + z\varkappa_{yo} + z^3\varkappa_{y1} = v_{o,y} + \tfrac{1}{2}w_{b,y}^2 - zw_{b,yy} - \frac{4z^3}{3h^2}w_{s,yy} \tag{15b}$$

$$\begin{aligned}\gamma_{xy} &= \gamma_{xyo} + 2\varkappa_{xyo} + z^3\varkappa_{xy1} \\ &= u_{o,y} + v_{o,x} + w_{b,x}w_{b,y} - 2zw_{b,xy} - \frac{8z^3}{3h^2}w_{s,xy} \\ &= \beta + \tau_b w_{b,x} - 2z\tau_{b,x} - \frac{8z^3}{3h^2}\tau_{s,x}\end{aligned} \tag{15c}$$

$$\gamma_{xz} = \left(1 - \frac{4z^2}{h^2}\right) w_{s,x} \tag{15d}$$

The advantage of the present HSDT is that, if higher-order terms are neglected, it leads to the classical lamination theory.

The constitutive relations for HSDT can be written as

$$\begin{Bmatrix} Nx \\ Ny \\ Nxy \\ Mx \\ My \\ Mxy \\ Px \\ Py \\ Pxy \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \gamma_{xyo} \\ \varkappa_{xo} \\ \varkappa_{yo} \\ \varkappa_{xyo} \\ \varkappa_{x1} \\ \varkappa_{y1} \\ \varkappa_{xy1} \end{Bmatrix} \tag{16}$$

where

$$E_{ij} = \frac{1}{4}\sum_{k=1}^{N} Q_{ij}^k(h_{k+1}^4 - h_k^4), \qquad i,j = 1,2,6$$

$$F_{ij} = \frac{1}{5}\sum_{k=1}^{N} Q_{ij}^k(h_{k+1}^5 - h_k^5)$$

$$H_{ij} = \frac{1}{7}\sum_{k=1}^{N} Q_{ij}^k(h_{k+1}^7 - h_k^7)$$

Now equating Ny, My, and Py to zero, Nx, Nxy, Mx, Mxy, Px, and Pxy can be written in terms of ε_{xo}, γ_{xyo}, $\varkappa_{xo}$, $\varkappa_{xyo}$, $\varkappa_{x1}$, and $\varkappa_{xy1}$ as follows:

$$\begin{bmatrix} Nx \\ Nxy \\ Mx \\ Mxy \\ Px \\ Pxy \end{bmatrix} = \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} & E_{11} & E_{16} \\ A_{16} & A_{66} & B_{16} & B_{66} & E_{16} & E_{66} \\ B_{11} & B_{16} & D_{11} & D_{16} & F_{11} & F_{16} \\ B_{16} & B_{66} & D_{16} & D_{66} & F_{16} & F_{66} \\ E_{11} & E_{16} & F_{11} & F_{16} & H_{11} & H_{16} \\ E_{16} & E_{66} & F_{16} & F_{66} & H_{16} & H_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_{xo} \\ \varkappa_{xyo} \\ \varkappa_{x1} \\ \varkappa_{xy1} \end{bmatrix} + \begin{bmatrix} A_{12} & B_{12} & E_{12} \\ A_{26} & B_{26} & E_{26} \\ B_{12} & D_{12} & F_{12} \\ B_{26} & D_{26} & F_{26} \\ E_{12} & F_{12} & H_{12} \\ E_{26} & F_{26} & H_{26} \end{bmatrix} \begin{bmatrix} X1 & X2 & X3 & X4 & X5 & X6 \\ Y1 & Y2 & Y3 & Y4 & Y5 & Y6 \\ Z1 & Z2 & Z3 & Z4 & Z5 & Z6 \end{bmatrix} \tag{17}$$

where $X1,X2,\ldots,X6$, $Y1,Y2,\ldots,Y6$, and $Z1,Z2,\ldots,Z6$ are given in the Appendix.

The constitutive relation after modification as in the case of the CLT and FSDT, that is by adding transverse shear stiffness terms in Eq. (17), becomes

$$\begin{Bmatrix} Nx \\ Nxy \\ Mx \\ Mxy \\ Px \\ Pxy \\ Q_x \end{Bmatrix} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} & \bar{D}_{14} & \bar{D}_{15} & \bar{D}_{16} & 0 \\ \bar{D}_{12} & \bar{D}_{22} & \bar{D}_{23} & \bar{D}_{24} & \bar{D}_{25} & \bar{D}_{26} & 0 \\ \bar{D}_{13} & \bar{D}_{23} & \bar{D}_{33} & \bar{D}_{34} & \bar{D}_{35} & \bar{D}_{36} & 0 \\ \bar{D}_{14} & \bar{D}_{24} & \bar{D}_{34} & \bar{D}_{44} & \bar{D}_{45} & \bar{D}_{46} & 0 \\ \bar{D}_{15} & \bar{D}_{25} & \bar{D}_{35} & \bar{D}_{45} & \bar{D}_{55} & \bar{D}_{56} & 0 \\ \bar{D}_{16} & \bar{D}_{26} & \bar{D}_{36} & \bar{D}_{46} & \bar{D}_{56} & \bar{D}_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{D}_{77} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_{xo} \\ \varkappa_{xyo} \\ \varkappa_{x1} \\ \varkappa_{xy1} \\ \gamma_{xz} \end{Bmatrix} \tag{18}$$

with

$$\bar{D}_{77} = \left(A_{44} - \frac{8}{h^2}D_{44} + \frac{16}{h^4}F_{44}\right)$$

$$A_{44} = \sum_{k=1}^{N} Q_{44}^k(h_{k+1} - h_k)$$

$$D_{44} = \frac{1}{3}\sum_{k=1}^{N} Q_{44}^k(h_{k+1}^3 - h_k^3)$$

$$F_{44} = \frac{1}{5}\sum_{k=1}^{N} Q_{44}^k(h_{k+1}^5 - h_k^5)$$

The field variables u_o, β, w_b, w_s, τ_b, and τ_s for the finite element formulation are assumed as

$$u_o = \sum_{i=1}^{4} N_i u_i, \qquad \beta = \sum_{i=1}^{4} N_i \beta_i \tag{19a}$$

$$w_b = \sum_{i=1}^{4} N_i w_{bi}, \qquad \tau_b = \sum_{i=1}^{4} N_i \tau_{bi} \tag{19b}$$

$$w_s = \sum_{i=1}^{4} N_i w_{si}, \qquad \tau_s = \sum_{i=1}^{4} N_i \tau_{si} \tag{19c}$$

Substituting Eq. (19) into the strain-displacement relations, Eq. (15), the strains can be written in terms of nodal degrees of freedom as

$$\begin{Bmatrix} \varepsilon_{xo} \\ \gamma_{xyo} \\ \varkappa_{xo} \\ \varkappa_{xyo} \\ \varkappa_{x1} \\ \varkappa_{xy1} \\ \gamma_{xz} \end{Bmatrix} = [B_o + \tfrac{1}{2}B_L(q_i)]\{q_i\} \tag{20}$$

with

$$\{q_i\} = [u_i\ u_{,xi}\ \beta_i\ \beta_{,xi}\ w_{bi}\ w_{b,xi}\ w_{si}\ w_{s,xi}\ \tau_{bi}\ \tau_{b,xi}\ \tau_{si}\ \tau_{s,xi}]$$

$$[B_o] = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 & 0 \\ 0 & 0 & -N_{i,xx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2N_{i,x} & 0 \\ 0 & 0 & 0 & -cN_{i,xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2cN_{i,x} \\ 0 & 0 & 0 & N_{i,x} & 0 & 0 \end{bmatrix}$$

$$[B_L] = \begin{bmatrix} 0 & 0 & w_{,x}N_{i,x} & 0 & 0 & 0 \\ 0 & 0 & \tau N_{i,x} & 0 & w_{,x}N_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The strain energy and kinetic energy of a composite beam can be written as

$$U = \frac{b}{2}\int_o^L \{q_i\}^T[B_o + \tfrac{1}{2}B_L(q_i)]^T[D][B_o + \tfrac{1}{2}B_L(q_i)]\{q_i\}\,dx \tag{21}$$

$$T(\text{CLT}) = \frac{b}{2}\int_o^L R\dot{w}^2\,dx \tag{22a}$$

$$T(\text{FSDT}) = \frac{b}{2}\int_o^L [R(\dot{w}_b + \dot{w}_s)^2 + I(\dot{\tau}^2 + \dot{w}_{b,x}^2)]\,dx \tag{22b}$$

$$T(\text{HSDT}) = \frac{b}{2}\int_o^L [R(\dot{w}_b + \dot{w}_s)^2 + I(\dot{\tau}^2 + \dot{w}_{b,x}^2) + c_2 I_7(\dot{w}_{s,x}^2 + \dot{\tau}_{s,x}^2) + 2cI_5(\dot{w}_{b,x}\dot{w}_{s,x} + \dot{\tau}_b\dot{\tau}_s)]\,dx \tag{22c}$$

(neglecting in-plane inertia) with $c = 4/3h^2$; $c_2 = 16/9h^4$; and $R, I, I_5, I_7 = \int_{-t/2}^{t/2} \rho_i\,(1,z^2,z^4,z^6)\,dz$.

Applying Hamilton's principle, the following dynamic finite element equations can be obtained:

$$\begin{bmatrix} k_{uu} & k_{uw} \\ k_{wu} & k_{ww} \end{bmatrix}\begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \bar{k}_{uw} \\ \bar{k}_{wu} & \bar{k}_{ww} \end{bmatrix}\begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 \\ 0 & \bar{\bar{k}}_{ww} \end{bmatrix}\begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \ddot{q}_u \\ \ddot{q}_w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{23}$$

where

$$\begin{bmatrix} k_{uu} & k_{uw} \\ k_{wu} & k_{ww} \end{bmatrix} = b\int_o^L [B_o][D][B_o]\,dx$$

$$\begin{bmatrix} 0 & \bar{k}_{uw} \\ \bar{k}_{wu} & \bar{k}_{ww} \end{bmatrix} = b\int_o^L (2[B_L][D][B_o] + [B_o][D][B_L])\,dx$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \bar{\bar{k}}_{ww} \end{bmatrix} = \frac{3}{2}b\int_o^L [B_L][D][B_L]\,dx$$

Assuming the harmonic oscillations assumption ($\ddot{q}_b = -\omega^2 q_b$), Eq. (23) is solved iteratively to obtain the linear and nonlinear frequencies and normalized spatial deformations. It may be observed that the axial and out-of-plane equations are exactly satisfied with the harmonic oscillation assumption; whereas in the case of Ref. 12, the out-of-plane equations with linear frequency and axial displacement computed using linear normalized spatial deformation will not get satisfied. In the present procedure, for simply supported uniform beams with immovable ends, the maximum axial displacement at quarter point of the beam in the converged vector exactly coincides with the continuum solution having a value $-\pi W_{max}/8l$, where W_{max} is the maximum transverse deflection. Also, for unsymmetric laminates the converged spatial deformations in the positive deflection half-cycle and negative deflection half-cycle will be different owing to the fact that the bending stiffness is direction dependent due to the presence of the bending-extension coupling. This situation is taken care of by reducing the partitioned matrix Eq. (23) to two second-order ordinary nonlinear differential equations: one for the positive deflection half-cycle and the other for the negative deflection half-cycle as follows:

$$\ddot{A} + \alpha_1 A + \beta_1 A^2 + \gamma_1 A^3 = 0 \tag{24a}$$

$$\ddot{A} + \alpha_2 A + \beta_2 A^2 + \gamma_2 A^3 = 0 \tag{24b}$$

where

$$\alpha_1 = \{q_b\}^T[M]^{-1}[k_{ww} - k_{wu}k_{uu}^{-1}k_{uw}]\{q_b\}$$

$$\beta_1 = \{q_b\}^T[M]^{-1}[\bar{k}_{ww} - k_{wu}k_{uu}^{-1}\bar{k}_{uw} - \bar{k}_{wu}k_{uu}^{-1}k_{uw}]\{q_b\}$$

$$\gamma_1 = \{q_b\}^T[M]^{-1}[\bar{\bar{k}}_{ww} - \bar{k}_{wu}k_{uu}^{-1}\bar{k}_{uw}]\{q_b\}$$

with $\{q_b\}$ being the converged spatial deformations in the positive deflection half-cycle. Similarly α_2, β_2, and γ_2 are evaluated using spatial deformations in the negative half-cycle.

Solving the Eq. (24) using the direct numerical integration method,[18,19] that is, premultiplying the equations with $\dot{A}$ and integrating with respect to time, leads to the following energy-balance equations:

$$\dot{A}^2 + \alpha_1 A^2 + \tfrac{2}{3}\beta_1 A^3 + \tfrac{1}{2}\gamma_1 A^4 = H = \text{const} \tag{25a}$$

$$\dot{A}^2 + \alpha_2 A^2 + \tfrac{2}{3}\beta_2 A^3 + \tfrac{1}{2}\gamma_2 A^4 = H = \text{const} \tag{25b}$$

The constant H is computed using the initial conditions, i.e., at $t = 0$, $A = A_{max}$, and $\dot{A} = 0$ as

$$H = \alpha_1 A_{max}^2 + \tfrac{2}{3}\beta_1 A_{max}^3 + \tfrac{1}{2}\gamma_1 A_{max}^4 \tag{26}$$

Substituting this energy constant H from Eq. (26) into Eq. (25b) leads to

$$\dot{A}^2 = H - \alpha_2 A^2 - \tfrac{2}{3}\beta_2 A^3 - \tfrac{1}{2}\gamma_2 A^4 \tag{27}$$

In the absence of coefficients β_1 and β_2, i.e., for isotropic, orthotropic, symmetric cross-ply, and symmetric and antisymmetric angle-ply laminates, Eq. (27) at the extreme position ($\dot{A} = 0$) will have two real equal and opposite roots ($\pm A_{max}$), but otherwise Eq. (27) will lead to two real, unequal, and opposite roots. In the present study, the opposite root $-B_{max}$ is computed using the Newton-Raphson method. To evaluate the nonlinear period/frequency of such laminates, the half-period of the positive deflection half-cycle and the other half-period of the negative deflection half-cycle are computed as follows:

$$T_{NL} = \frac{2\pi}{\omega} = 2\left\{\int_o^{A_{\max}} \frac{dA}{[\alpha_1(A_{\max}^2 - A^2) + \frac{2}{3}\beta_1(A_{\max}^3 - A^3) + \frac{1}{2}\gamma_1(A_{\max}^4 - A^4)]^{1/2}} + \int_o^{-B_{\max}} \frac{dA}{[\alpha_2(B_{\max}^2 - A^2) - \frac{2}{3}\beta_2(B_{\max}^3 + A^3) + \frac{1}{2}\gamma_2(B_{\max}^4 - A^4)]^{1/2}}\right\} \tag{28}$$

Substituting $A = A_{\max}\sin\theta$ in the first integral and $A = -B_{\max}\sin\theta$ in the second integral leads to

$$T_{NL} = 2\left\{\int_o^{\pi/2} \frac{d\theta}{\left[\alpha_1\left(1 + \frac{2}{3}\frac{\beta_1}{\alpha_1}\frac{1+\sin\theta+\sin^2\theta}{1+\sin\theta}A_{\max} + \frac{1}{2}\frac{\gamma_1}{\alpha_1}(1+\sin^2\theta)A_{\max}^2\right)\right]^{1/2}} + \int_o^{\pi/2} \frac{d\theta}{\left[\alpha_2\left(1 - \frac{2}{3}\frac{\beta_2}{\alpha_2}\frac{1+\sin\theta+\sin^2\theta}{1+\sin\theta}B_{\max} + \frac{1}{2}\frac{\gamma_2}{\alpha_2}(1+\sin^2\theta)B_{\max}^2\right)\right]^{1/2}}\right\} \tag{29}$$

The integrands in the present paper are evaluated using a five-point Gaussian quadrature. It may be observed that the solution will correspond to that of Woinowsky Krieger[23] for isotropic beams, where $\beta_1 = \beta_2 = 0$. Furthermore, in the present scheme no solution need to be assumed in time.

Numerical Results and Discussions

Nonlinear vibration behavior of unsymmetric composite beams with immovable end conditions is obtained using the procedure described in the previous section. The nonlinear frequency ratios are computed for two slenderness ratios, 200 and 25, to show the effectiveness of various theories such as the CLT, FSDT, and HSDT. The frequency ratios of isotropic, orthotropic, and symmetric laminates are computed and found to agree very well with those of Ref. 12 and hence not presented herein. Throughout this section, results are presented for two-layered cross-ply (0 deg/90 deg), two-layered angle-ply (45 deg/−45 deg), and four-layered (0 deg/45 deg/−45 deg/90 deg) laminates.

The mechanical properties assumed for a graphite-epoxy, unidirectional laminate are $E_L = 18.5 \times 10^6$ psi (1.301×10^6 ksc), $E_T = 1.6 \times 10^6$ psi (1.125×10^5 ksc), $G_{LT} = 0.65 \times 10^6$ psi (4.571×10^4 ksc), $G_{LZ} = G_{LT}$, $\nu_{LT} = 0.25$, and $\rho = 0.055$ lb/in.^{3} (4.768×10^{-5} kg/cm^3).

The boundary conditions used for the beam elements based on the CLT, FSDT, and HSDT are

Hinged end (at $x = 0$ or 1):

$u = w = \tau = 0$ (CLT)

$u = w_b = w_s = \tau = 0$ (FSDT)

$u = w_b = w_s = \tau_b = \tau_s = 0$ (HSDT)

Fixed End (at $x = 0$ or 1):

$u = w = w' = \tau = 0$ (CLT)

$u = w_b = w_b' = w_s = w_s' = \tau = 0$ (FSDT)

$u = w_b = w_b' = w_s = w_s' = \tau_b = \tau_s = 0$ (HSDT)

Table 1 gives the comparison of linear frequencies for the first four modes for a fixed-free, 30-deg laminate, obtained using CLT, FSDT, HSDT, and those of Kapania and Raciti[12] and Abarcar and Cuniff.[7] It is clear that the CLT results are in very good agreement with those obtained from the FSDT, HSDT, and others. The CLT always gives upper bound of the frequencies can also be seen. Little difference in the present FSDT and that of Kapania and Raciti[12] can be attributed to the numerical procedures and the computer used in the two computations. Furthermore, Kapania and Raciti[12] have included a term $(hb^3/12)\dot{\tau}^2$ in the kinetic energy which is not included herein, because this term has no justification until the displacement field assumed is changed in such a way that $w = w_b + w_s + y\,\tau$. In that case, the kinetic energy will have a term $J\dot{\tau}^2(J = hb^3/12 + bh^3/12)$ and, at the same time, the stiffness coefficient bD_{66} in the strain energy gets modified to $b(D_{66} + A_{44}b^2/12)$ leading to very high frequencies. The reason for this could be that, for torsional vibrations, the torsional constant $J = bh^3/12$ is not the polar moment of inertia. Since, in the present paper, the authors intend to investigate only the flexural vibrations and the effect of bending-extension and bending-twisting coupling and not the torsional vibrations, all of the above suggested modifications are not affected herein.

The linear frequency parameter λ_ω for beams having lay-ups 45 deg/−45 deg, 0 deg/90 deg, and 0 deg/45 deg/−45 deg/90 deg and slenderness ratios of 200 and 25 under hinged-hinged, fixed-hinged, and fixed-fixed boundary conditions are presented in Table 2. It may be observed that for the four-layered beam, HSDT results differ from those of the CLT and FSDT even for large slenderness ratios, thus demonstrating the effect of transverse shear. The frequency parameter λ_ω for beams having a slenderness ratio of 200 obtained using the CLT and FSDT is nearly same, whereas in the case of short beams (slenderness ratio = 25), the FSDT and HSDT results are close compared to CLT.

The nonlinear frequency ratios at various amplitude ratios for two-layered cross-ply (0 deg/90 deg), two-layered angle-

Table 1 Comparison of linear frequencies of an orthotropic beam[a]

Mode no.	Frequency ω_o, Hz: Present, CLT	Present, FSDT	Present, HSDT	Ref. 12	Ref. 7, Theory	Ref. 7, Experimental
1	52.6018	52.5846	52.5817	52.6500	52.7000	52.7000
2	330.093	329.363	329.241	329.780	329.000	331.800
3	932.414	927.504	926.683	928.290	915.900	924.700
4	1839.79	1822.061	1819.08	1818.42	1896.50	1827.40

[a]$E_L = 18.74 \times 10^6$ psi (1.318×10^6 ksc); $E_T = 1.367 \times 10^6$ psi (0.961×10^5 ksc); $G_{LT} = 0.7479 \times 10^6$ psi (5.259×10^4 ksc); $\nu_{LT} = 0.3$; $G_{LZ} = 0.6242 \times 10^6$ psi (4.390×10^4 ksc); $\rho = 0.055$ lb/in.^{3} (4.768×10^{-5} kg/cm^3); $l = 7.5$ in. (19.05 cm); $b = 0.5$ in. (1.27 cm); and $t = 0.125$ in. (0.3175 cm).

Table 2 Linear frequency parameter for various lay-ups using CLT, FSDT, and HSDT

		Frequency parameter λ_ω $(\rho A\omega_0^2 l^4/E_1 I)$					
		Slenderness ratio = 200			Slenderness ratio = 25		
Boundary condition	Theory	45 deg/−45 deg	0 deg/90 deg	0 deg/45 deg/−45 deg/90 deg	45 deg/−45 deg	0 deg/90 deg	0 deg/45 deg/−45 deg/90 deg
Hinged-hinged	CLT	10.84	43.69	41.24	10.84	43.69	41.24
	FSDT	10.83	43.53	41.09	10.23	35.19	33.57
	HSDT	10.62	43.51	40.33	9.92	34.94	32.99
Fixed-hinged	CLT	26.47	69.20	68.55	26.47	69.20	68.55
	FSDT	26.42	68.82	68.17	23.32	51.02	50.65
	HSDT	25.92	68.84	65.36	22.66	51.73	49.18
Fixed-fixed	CLT	55.83	128.38	129.71	55.83	128.38	129.71
	FSDT	55.59	127.14	128.44	43.91	78.98	79.45
	HSDT	54.54	127.22	122.05	42.95	82.03	78.23

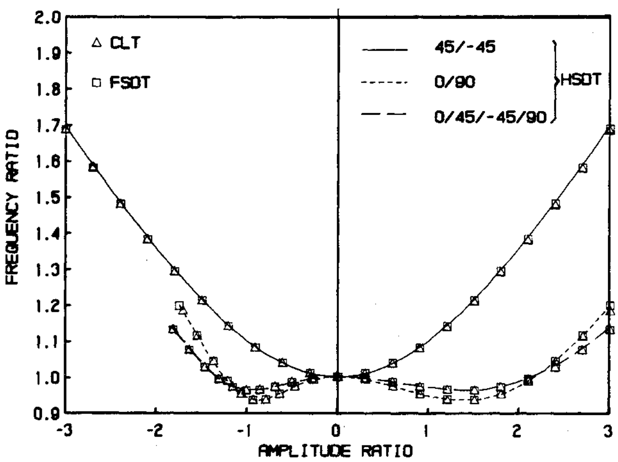

Fig. 2 Variations of frequency ratio with amplitude ratio of hinged-hinged beam (slenderness ratio = 200).

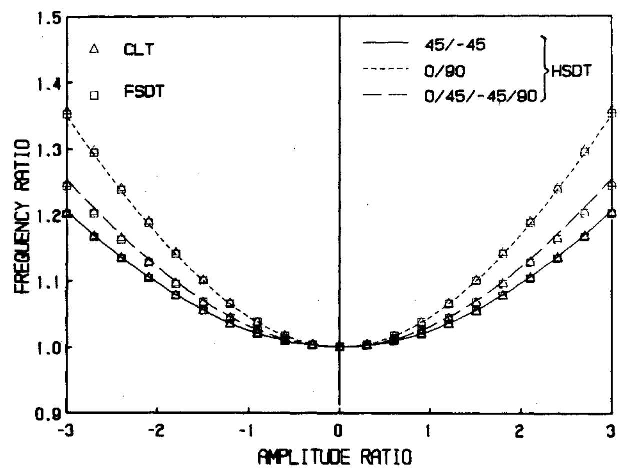

Fig. 4 Variations of frequency ratio with amplitude ratio of fixed-fixed beam (slenderness ratio = 200).

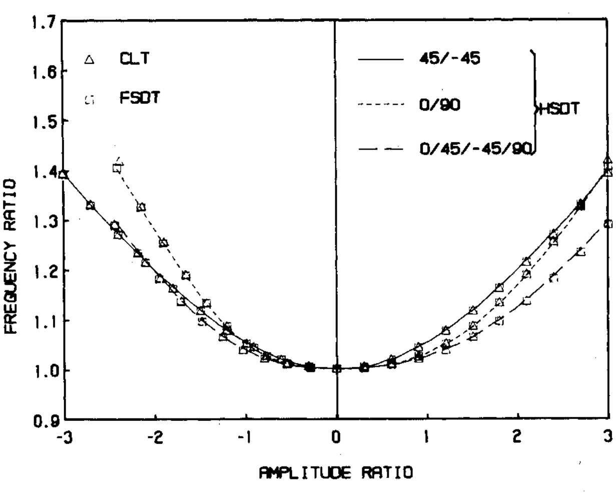

Fig. 3 Variations of frequency ratio with amplitude ratio of fixed-hinged beam (slenderness ratio = 200).

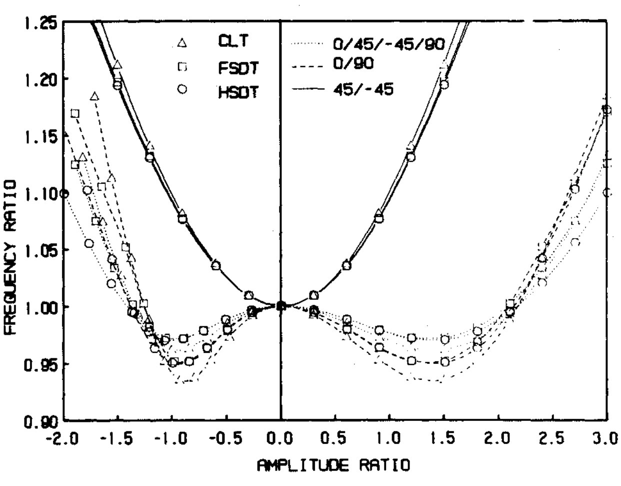

Fig. 5 Variations of frequency ratio with amplitude ratio of hinged-hinged beam (slenderness ratio = 25).

ply (45 deg/−45 deg), and four-layered (0 deg/45 deg/−45 deg/90 deg) laminates having a slenderness ratio of 200, under hinged-hinged conditions, using the CLT, FSDT, and HSDT are presented in Fig. 2. It may be observed that all of the three theories yield nearly the same frequencies and frequency ratios. It is interesting to note that for the two-layered cross-ply (0 deg/90 deg) and four-layered (0 deg/45 deg/−45 deg/90 deg) laminates, the frequency ratio vs amplitude-ratio curves are not symmetric about the frequency ratio axis, whereas for the two-layered angle-ply laminate the curve is symmetric. It may also be seen that for the above two types of laminates, the nonlinearity is of the soft-spring type for small amplitudes and of the hard spring-type for large amplitudes. Hence, laminates having bending-extension coupling of B_{11} type have different amplitudes of the positive deflection half-cycle and negative half-cycle, softening type of nonlinearity for small amplitudes, and hardening type for large amplitudes.

Figures 3 and 4 show the same variation for the fixed-hinged and fixed-fixed beams. It can be noticed that for the beams having the fixed-hinged condition, the nonlinearity is always

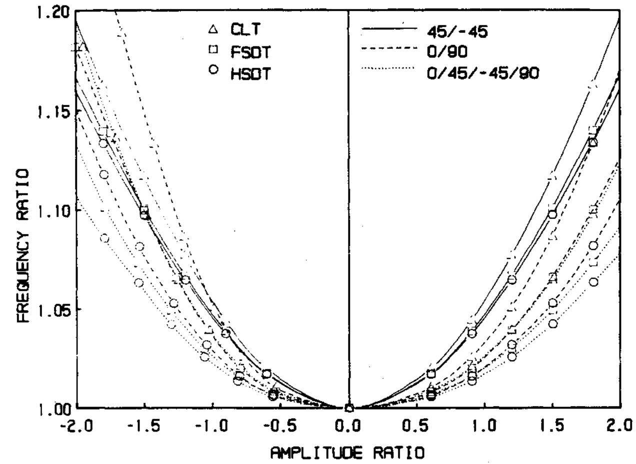

Fig. 6 Variations of frequency ratio with amplitude ratio of fixed-hinged beam (slenderness ratio = 25).

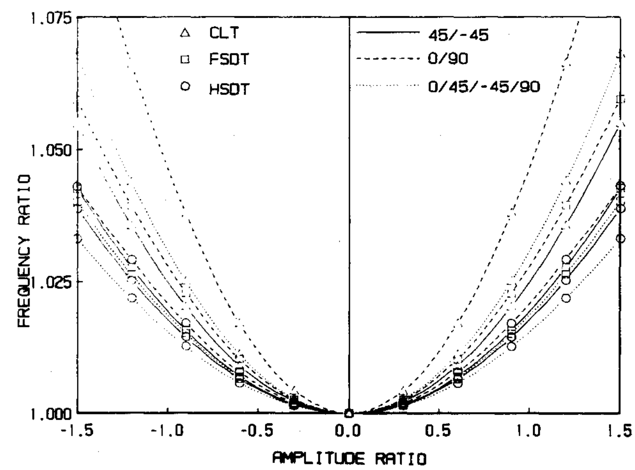

Fig. 7 Variations of frequency ratio with amplitude ratio of fixed-fixed beam (slenderness ratio = 25).

of the hardening type and the lay-ups having bending-extension coupling (B_{11}) yield a different amplitude for the positive and negative deflection half-cycles. For fixed-fixed beams again, the nonlinearity is always of the hardening type and the frequency-ratio vs amplitude-ratio curve is always symmetric about frequency ratio axis. Hence, for the fixed-fixed beams, the amplitude of the positive deflection half-cycle and the negative deflection half-cycle is always the same for unsymmetric laminates as well. Similarly, for antisymmetric angle-ply laminates, the amplitude of two half-cycles remains the same for all of the boundary conditions considered. It may be noted that the results of Kapania and Raciti[12] are incorrect, because of the use of the perturbation method, which the authors have already proved in their earlier works.[18,19]

Figures 5–7 give the variation of frequency ratios with amplitude ratios for 0 deg/90 deg, 45 deg/−45 deg, and 0 deg/45 deg/−45 deg/90 deg laminates having a slenderness ratio of 25, under hinged-hinged, hinged-fixed, and fixed-fixed conditions, using the CLT, FSDT, and HSDT. Apart from the conclusions drawn from Figs. 2–4, it is observed that the FSDT and HSDT results agree very well for the 45 deg/−45 deg laminate for all of the three boundary conditions considered. For the other two lay-ups considered, the frequency ratios obtained from the FSDT do not match well with those of the HSDT. It is also to be noted that the CLT results for hinged-fixed and fixed-fixed beams when compared with the FSDT and HSDT results are far off at large amplitudes.

Conclusions

The direct numerical integration method results in very accurate predictions of nonlinear frequencies of composite laminates. The perturbation method for unsymmetric laminates, wherein the coefficient of quadratic term β is present, leads to incorrect results. The bending stiffness of unsymmetric laminates is direction dependent, leading to different spatial deformations in the positive deflection and negative deflection half-cycles; consequently, the magnitude of β for two half-cycles is different. The nonlinearity for such laminates at small amplitudes is of softening type; and at large amplitude, it is of the hardening type for hinged-hinged beams. However, for the fixed-hinged and fixed-fixed beams, the nonlinearity is always of the hardening type. The frequency-ratio vs amplitude-ratio curve for unsymmetric laminates where bending-extension coupling (B_{11}) is predominant is not symmetric about the frequency ratio axis.

The CLT, FSDT, and HSDT yield nearly the same fundamental frequencies and frequency ratios for beams having large slenderness ratios; but, for small slenderness ratios, the results of the CLT and FSDT do not agree very well with the higher-order shear-deformable theory (HSDT). Furthermore, the HSDT proposed herein has CLT as a subset.

The results obtained from the CLT are accurate enough for large slenderness ratios; and, hence, one need not use shear deformable theories for such beams. For low l/r, the HSDT is recommended for obtaining accurate results.

Appendix

The coefficients $X1, X2, \ldots, X6, Y1, Y2, \ldots, Y6$, and $Z1, Z2, \ldots, Z6$ are as follows:

$$c1 = \left[F_{22} - \frac{B_{22}E_{22}}{A_{22}}\right] \Big/ \left[H_{22} - \frac{E_{22}^2}{A_{22}}\right]$$

$$c2 = \left[F_{22} - \frac{B_{22}E_{22}}{A_{22}}\right] \Big/ \left[D_{22} - \frac{B_{22}^2}{A_{22}}\right]$$

$$c3 = \left[D_{22} - \frac{B_{22}^2}{A_{22}}\right] - \frac{\left[F_{22} - \frac{B_{22}E_{22}}{A_{22}}\right]^2}{\left[H_{22} - \frac{E_{22}E_{22}}{A_{22}}\right]}$$

$$c4 = \left[H_{22} - \frac{E_{22}^2}{A_{22}}\right] - \frac{\left[F_{22} - \frac{B_{22}E_{22}}{A_{22}}\right]^2}{\left[D_{22} - \frac{B_{22}B_{22}}{A_{22}}\right]}$$

$$Y1 = \left[\left(E_{12} - \frac{E_{22}A_{21}}{A_{22}}\right) c1 - \left(B_{12} - \frac{B_{22}A_{21}}{A_{22}}\right)\right] \Big/ c3$$

$$Y2 = \left[\left(E_{26} - \frac{E_{22}A_{26}}{A_{22}}\right) c1 - \left(B_{26} - \frac{B_{22}A_{26}}{A_{22}}\right)\right] \Big/ c3$$

$$Y3 = \left[\left(F_{12} - \frac{E_{22}B_{22}}{A_{22}}\right) c1 - \left(D_{12} - \frac{B_{22}B_{21}}{A_{22}}\right)\right] \Big/ c3$$

$$Y4 = \left[\left(F_{26} - \frac{E_{22}B_{26}}{A_{22}}\right) c1 - \left(D_{26} - \frac{B_{22}B_{26}}{A_{22}}\right)\right] \Big/ c3$$

$$Y5 = \left[\left(H_{12} - \frac{E_{22}E_{21}}{A_{22}}\right) c1 - \left(F_{12} - \frac{B_{22}E_{21}}{A_{22}}\right)\right] \Big/ c3$$

$$Y6 = \left[\left(H_{26} - \frac{E_{22}E_{26}}{A_{22}}\right) c1 - \left(F_{26} - \frac{B_{22}E_{26}}{A_{22}}\right)\right] \Big/ c3$$

$$Z1 = \left[\left(B_{12} - \frac{B_{22}A_{21}}{A_{22}}\right) c2 - \left(E_{12} - \frac{E_{22}A_{21}}{A_{22}}\right)\right] \Big/ c4$$

$$Z2 = \left[\left(B_{26} - \frac{B_{22}A_{26}}{A_{22}}\right) c2 - \left(E_{26} - \frac{E_{22}A_{26}}{A_{22}}\right)\right] \Big/ c4$$

$$Z3 = \left[\left(D_{12} - \frac{B_{22}B_{21}}{A_{22}}\right) c2 - \left(F_{12} - \frac{E_{22}B_{21}}{A_{22}}\right)\right] \Big/ c4$$

$$Z4 = \left[\left(D_{26} - \frac{B_{22}B_{26}}{A_{22}}\right) c2 - \left(F_{26} - \frac{E_{22}B_{26}}{A_{22}}\right)\right] \Big/ c4$$

$$Z5 = \left[\left(F_{12} - \frac{B_{22}E_{21}}{A_{22}}\right) c2 - \left(H_{12} - \frac{E_{22}E_{21}}{A_{22}}\right)\right] \Big/ c4$$

$$Z6 = \left[\left(F_{26} - \frac{B_{22}E_{26}}{A_{22}}\right) c2 - \left(H_{26} - \frac{E_{22}E_{26}}{A_{22}}\right)\right] \Big/ c4$$

$$X1 = -\frac{A_{12}}{A_{22}} - \frac{B_{22}}{A_{22}} Y1 - \frac{E_{22}}{A_{22}} Z1$$

$$X2 = -\frac{A_{26}}{A_{22}} - \frac{B_{22}}{A_{22}} Y2 - \frac{E_{22}}{A_{22}} Z2$$

$$X3 = -\frac{B_{12}}{A_{22}} - \frac{B_{22}}{A_{22}} Y3 - \frac{E_{22}}{A_{22}} Z3$$

$$X4 = -\frac{B_{26}}{A_{22}} - \frac{B_{22}}{A_{22}} Y4 - \frac{E_{22}}{A_{22}} Z4$$

$$X5 = -\frac{E_{12}}{A_{22}} - \frac{B_{22}}{A_{22}} Y5 - \frac{E_{22}}{A_{22}} Z5$$

$$X6 = -\frac{E_{26}}{A_{22}} - \frac{B_{22}}{A_{22}} Y6 - \frac{E_{22}}{A_{22}} Z6$$

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